Weakly Non Ergodic Noise

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AR EB **PRL 99** 208302 (2007)
SB EB **PRL 98** 250601 (2007)
EB **JSP 123** 883 (2006)
GM EB **PRL 90**, 104101 (2005)
GB EB **PRL 94**, 240602 (2005)

Outline

- Blinking Quantum Dots, Sub-Diffusion in Actin Networks, mRNA sub-diffusing in a cell.
- Anomalous Diffusion, power law waiting times
- Distribution of Time Averages for Weak Ergodicity Breaking (Bouchaud).
- Possible generalization of Boltzmann--Gibbs statistics.

Main Result: DISTRIBUTION of TIME AVERAGES

- Let $\mathcal{O}$ be a physical observable, and $\overline{\mathcal{O}}$ its time average,

$$f_\alpha\left(\overline{\mathcal{O}}\right) = -\frac{1}{\pi}\lim_{\epsilon\to 0}\mathrm{Im}\frac{\sum_{x=1}^{L} P_x^{eq}\left(\overline{\mathcal{O}} - \mathcal{O}_x + i\epsilon\right)^{\alpha-1}}{\sum_{x=1}^{L} P_x^{eq}\left(\overline{\mathcal{O}} - \mathcal{O}_x + i\epsilon\right)^{\alpha}}.$$

- Weak Ergodicity Breaking is related to Anomalous Diffusion

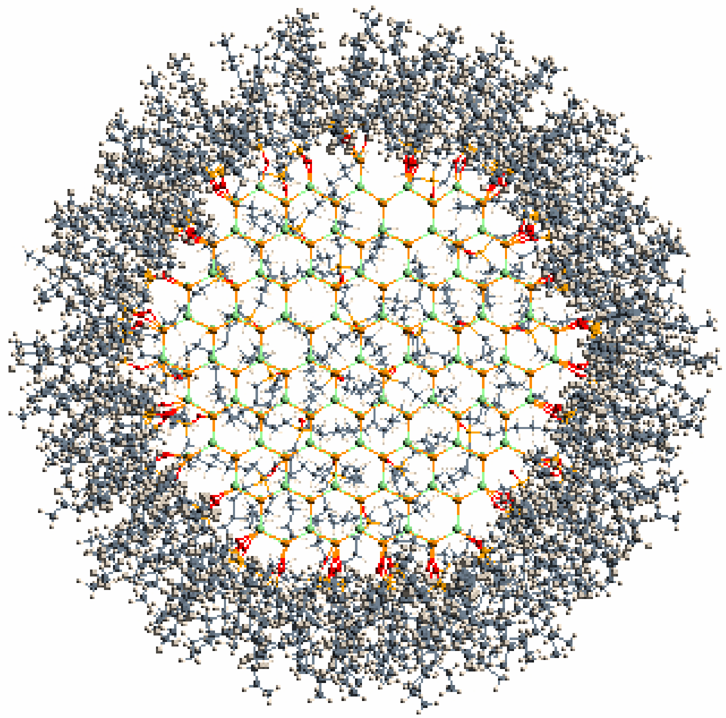
$$\langle x^2\rangle \sim t^\alpha \qquad 0<\alpha<1.$$

- Ergodic case $\alpha = 1$

$$f_1\left(\overline{\mathcal{O}}\right) = \delta\left(\overline{\mathcal{O}} - \langle\mathcal{O}\rangle\right).$$

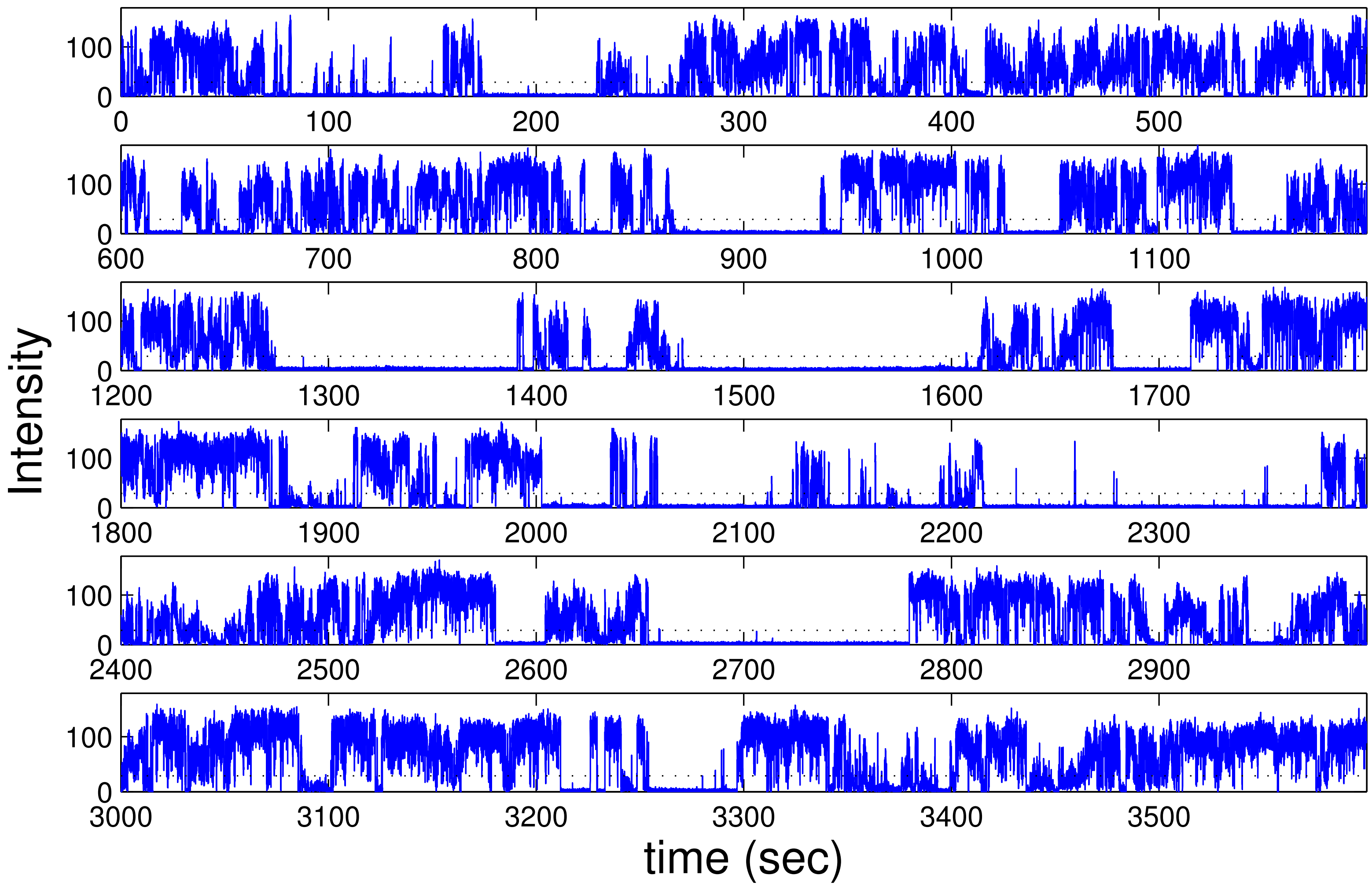
Rebenshtok, Barkai **PRL 99** 210601 (2007)

Nano Crystals also called Quantum Dots

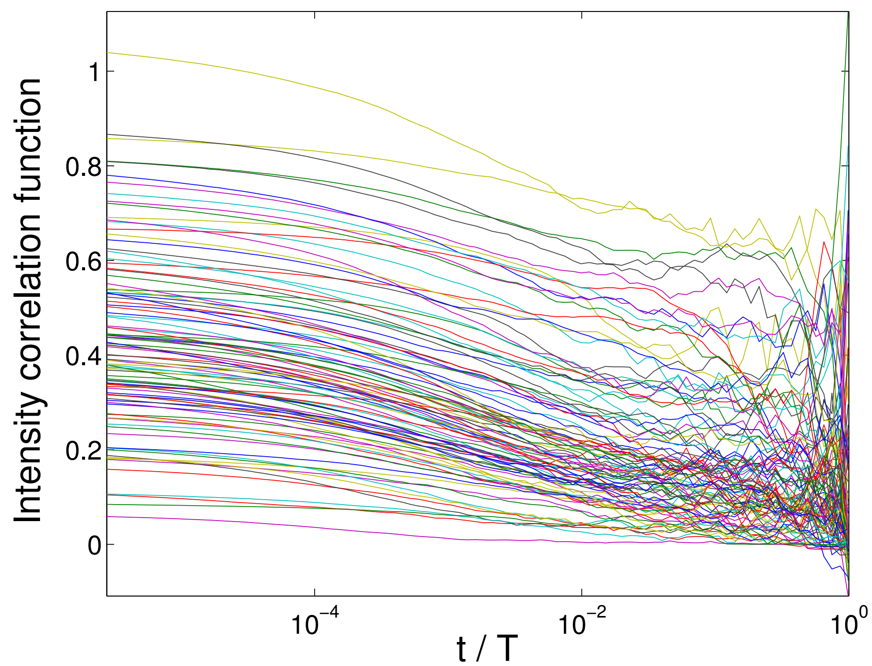


Grey - C, White - H, Orange - P, Red - O, Green - Cd, Gold - Se

Blinking Nano Crystals (coated CdSe)



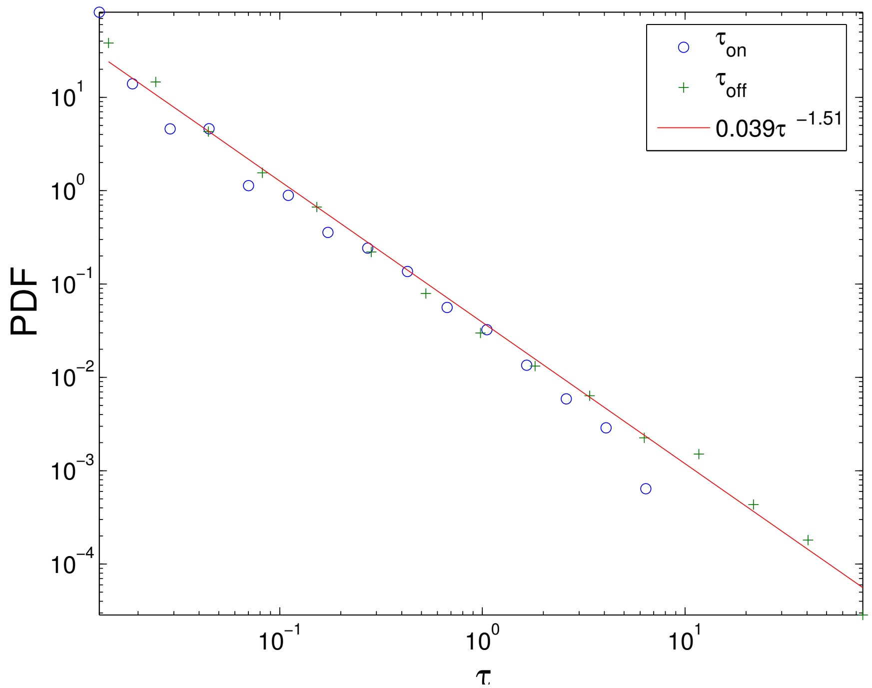
Non-ergodic Intensity Correlation Functions



Experiment Brokmann et al **Phys. Rev. Lett.** (2003).

Theory: Margolin, Barkai **Phys. Rev. Lett.** 90, 104101 (2005).

Power Law Distribution of on and Off times

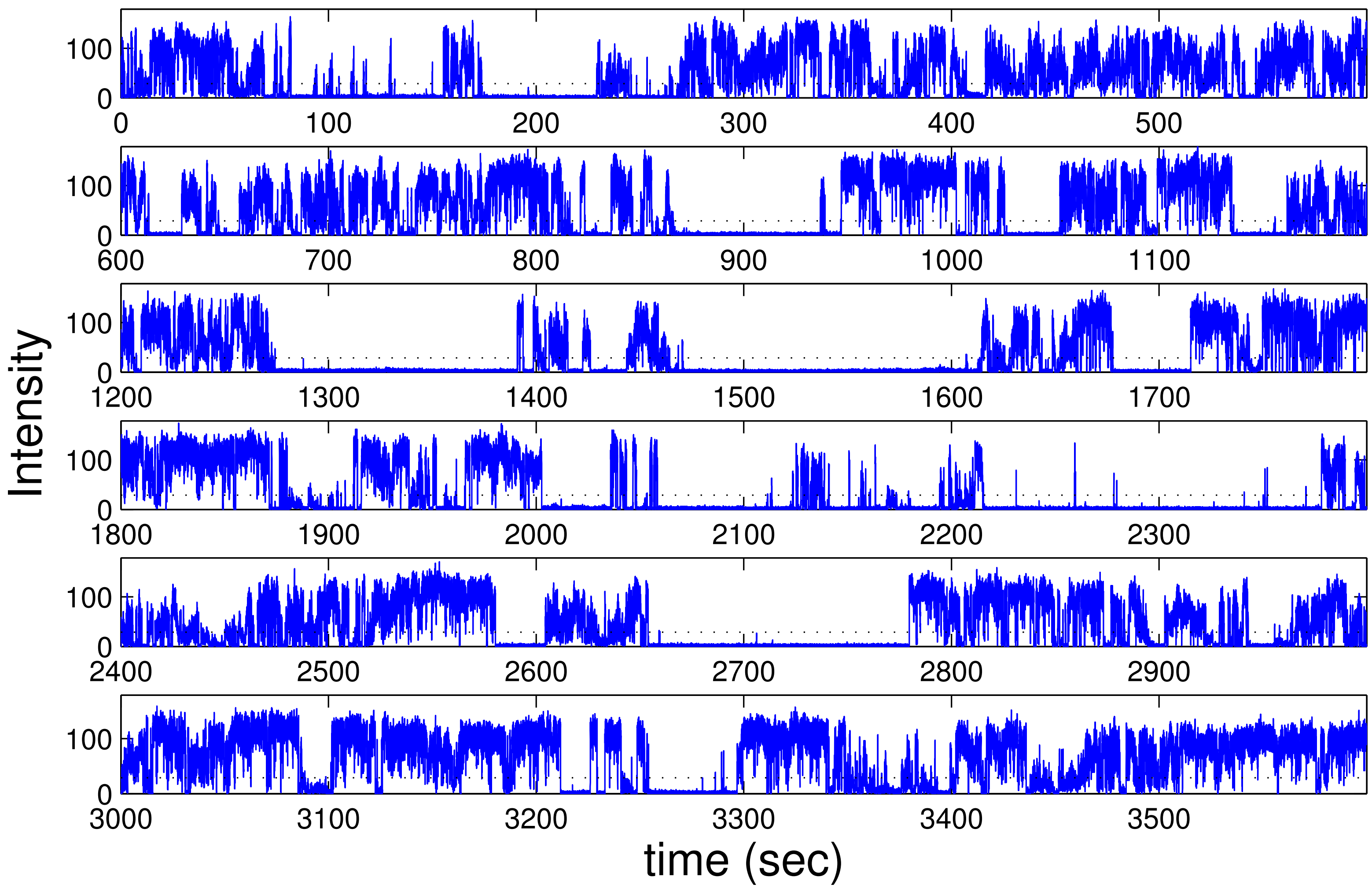


Averaged time in States On and Off is infinite.

If $I(t) < I_{\text{thresh}}$ process is in state off.

Power law waiting time $\psi(\tau) \propto \tau^{-(1+\alpha_{off})}$.

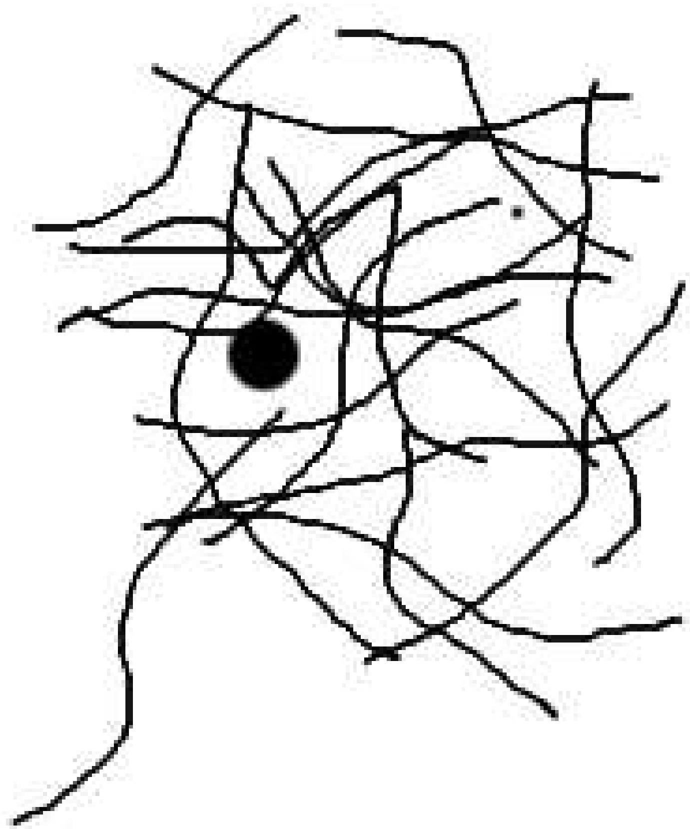
Power Law Sojourn Times Lead To Ergodicity Breaking and Non-Stationarity

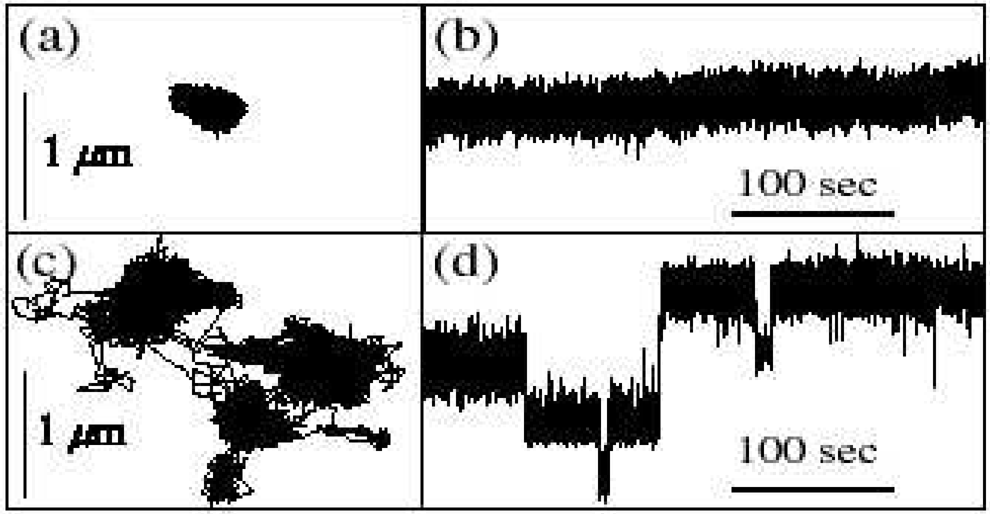


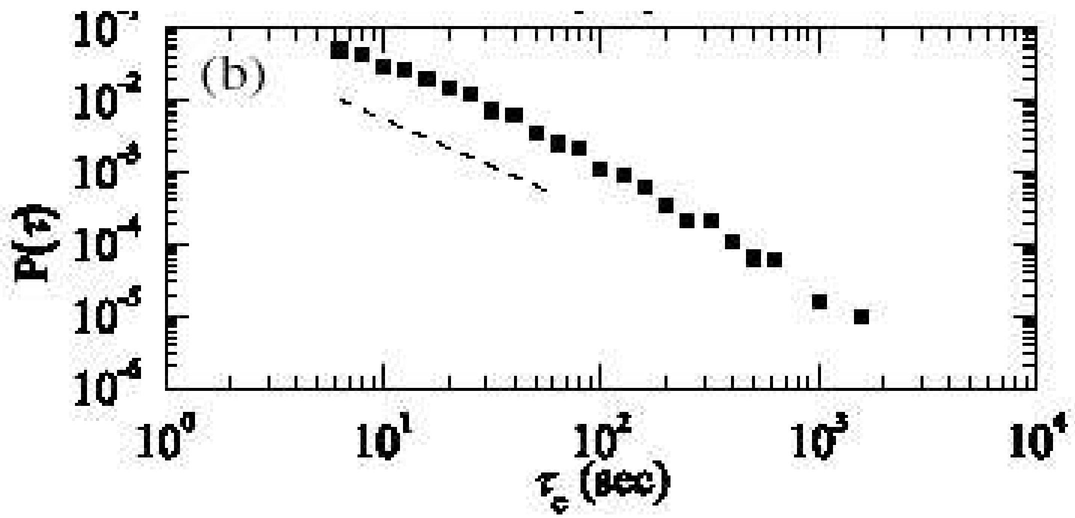
Continuous Time Random Walk (CTRW)

Dispersive Transport in Amorphous Material Scher-Montroll (1975).

Bead Diffusing in Polymer Network Weitz (2004).

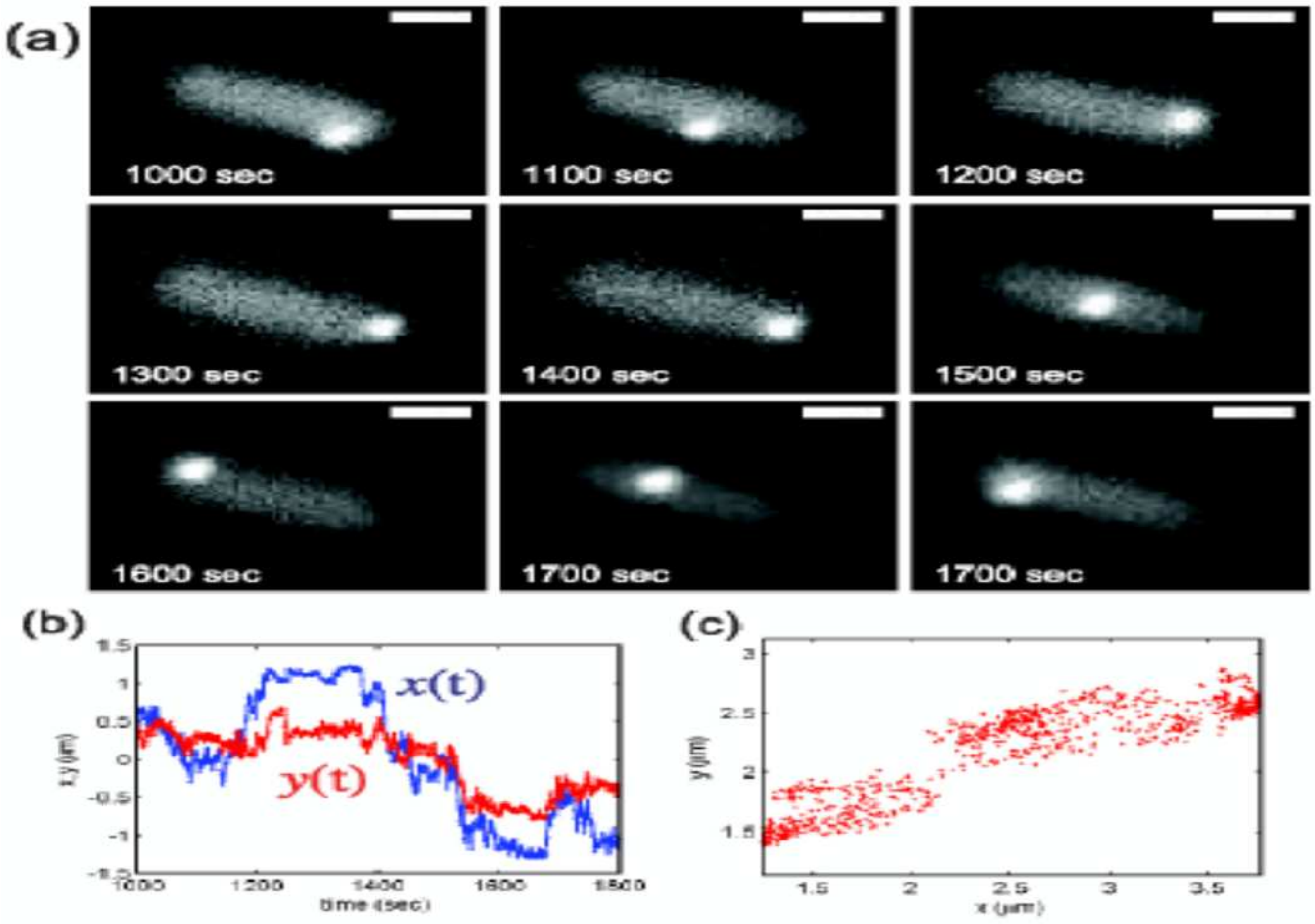


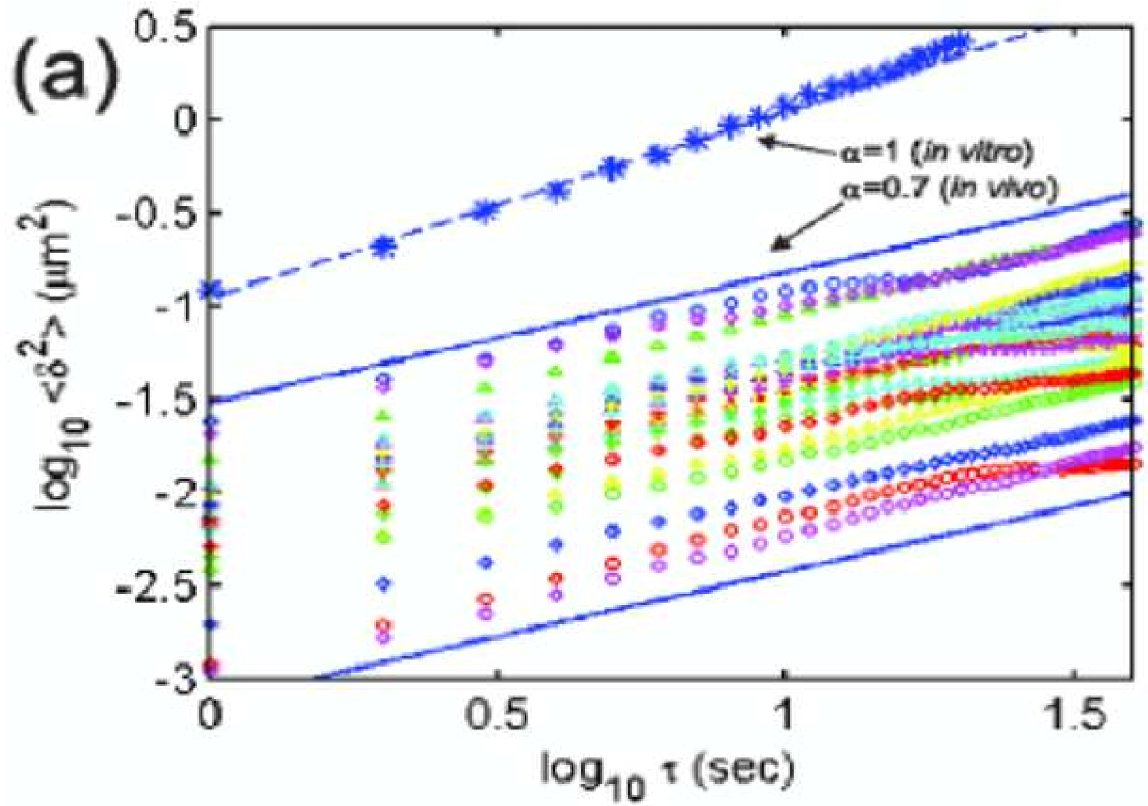




Average Waiting Time is ∞. Diffusion is anomalous $\langle r^2 \rangle \propto t^{\alpha}$.

mRNA diffusing in a cell (Goding Cox)




(a)
α=1 (in vitro)
α=0.7 (in vivo)
$\log_{10} <\delta^2> (\mu m^2)$
$\log_{10} \tau$ (sec)
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
0
0.5
1
1.5

Power Law Waiting Times Leads to Weak Ergodicity Breaking

Several other single particle experiments exhibit power law waiting time in a micro-state of the system

In single particle experiments the problem of ensemble averaging is removed.

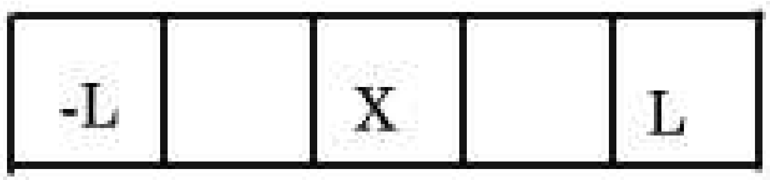
Time averages are not identical to ensemble averages, when the average waiting time is infinite.

What theory replaces ergodic statistical mechanics for such systems?

CTRW

- Renewal type of Random walk on lattice.
- Jumps to nearest neighbors only.
- q_x $(1 - q_x)$ Prob. of jumping from x to $x-1$ $(x+1)$.
- waiting times are i.i.d r.v with pdf

$$\psi(t) \propto t^{-1-\alpha} \qquad 0 < \alpha < 1$$



Time Averages

- Occupation fraction

$$\overline{p}_x = \frac{t_x}{t}.$$

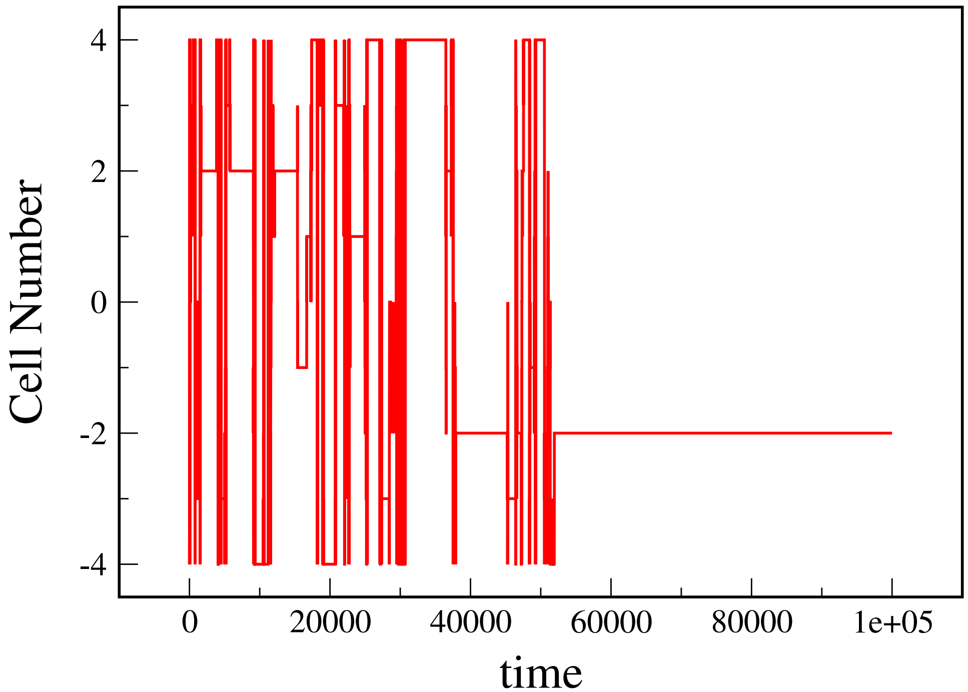
- Time average:

$$\overline{\mathcal{O}} = \sum_{x=-L,L} \mathcal{O}_x \overline{p}_x$$

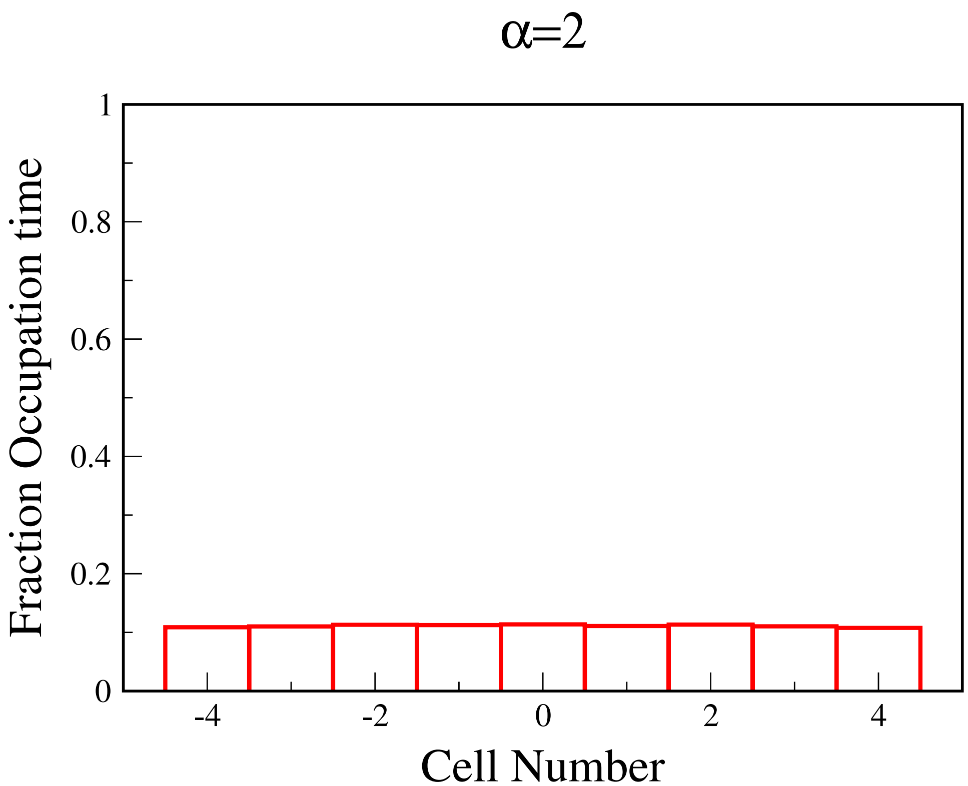
- For example

$$\overline{X} = \sum_{x=-L}^{L} x \overline{p}_x.$$

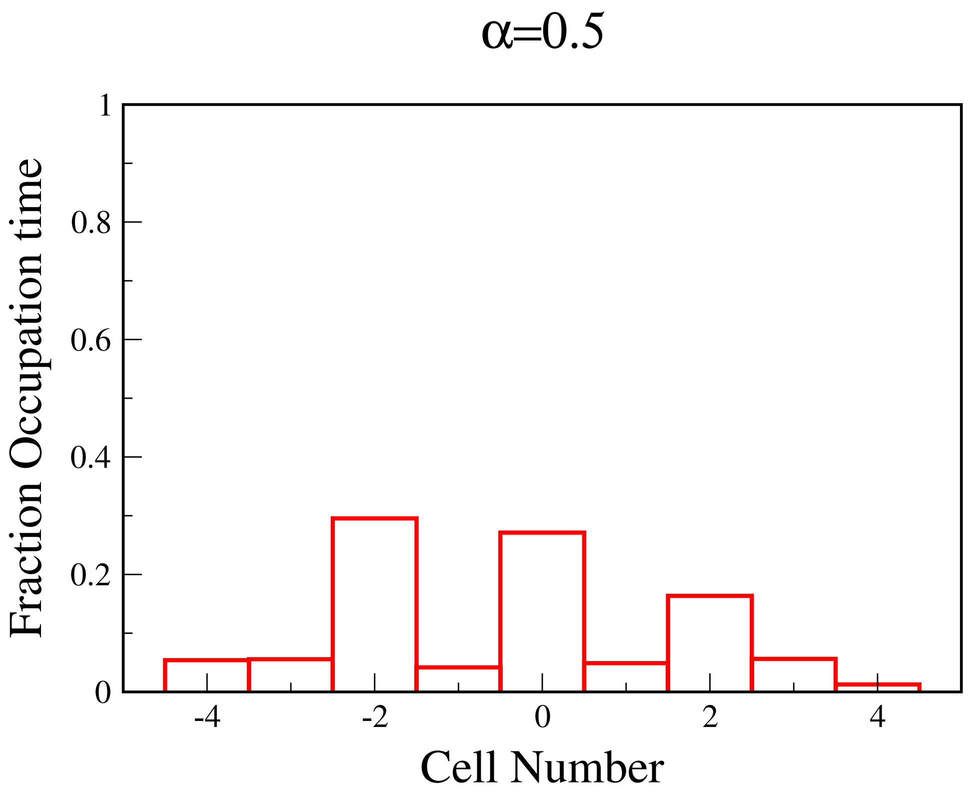
Trajectory Unbiased RW $q = 1/2$ $\alpha = 1/2$



Ergodic Phase



Non-Ergodic Phase



Question

Prob. that a member of an ensemble occupies lattice point x is P_x^{eq}

For an ergodic process $\overline{p}_x = P_x^{eq}$.

$\langle \mathcal{O} \rangle = \sum P_x^{eq} \mathcal{O}_x = \overline{\mathcal{O}}$

For $0 < \alpha < 1$ what is the pdf of $\overline{\mathcal{O}}$?

Two Ensembles

Fix n total number of jumps (t fluctuating)

Fix measurement time t (n fluctuating).

Population Dynamics in Discrete Time

$$P_x\left(n+1\right) = q_{x+1} P_{x+1}\left(n\right) + \left(1 - q_{x-1}\right) P_{x-1}\left(n\right).$$

When $n \to \infty$, an equilibrium is obtained $P_x^{eq}(n+1) = P_x^{eq}(n)$

Levy Statistics

- n_x number of times particle visits site x.
- When $n \to \infty$ $n_x/n = P_x^{eq}$
- t_x total time spent in state x. Sum i.i.d r.v. whose mean is infinite.
- Apply Lévy's limit theorem

$$f(t_x) = l_{\alpha, A_\alpha P_x^{eq} n}(t_x) \, .$$

- Use

$$\overline{p}_x = \frac{t_x}{\sum_{x=-L}^{L} t_x}$$

Multidimensional PDF of $\overline{p}_1, \cdots \overline{p}_L$

$$P_L\left(\overline{p}_1, \cdots, \overline{p}_L\right) = \delta\left(1 - \sum_{x=1}^{L} \overline{p}_x\right) \int_0^\infty dy y^{L-1} \Pi_{x=1}^{L} l_{\alpha, P_x^{eq}}\left(y \overline{p}_x\right)$$

Independent of n, details of $\psi(t)$.

Depends on α and P_x^{eq}.

The PDF of TIME AVERAGES

Using $\overline{\mathcal{O}} = \sum_x \overline{p}_x \mathcal{O}_x$ and the multi-dimensional joint PDF

$$f_\alpha\left(\overline{\mathcal{O}}\right) = -\frac{1}{\pi}\lim_{\epsilon\to 0}\text{Im}\frac{\sum_{x=1}^{L} P_x^{eq}\left(\overline{\mathcal{O}} - \mathcal{O}_x + i\epsilon\right)^{\alpha-1}}{\sum_{x=1}^{L} P_x^{eq}\left(\overline{\mathcal{O}} - \mathcal{O}_x + i\epsilon\right)^{\alpha}}.$$

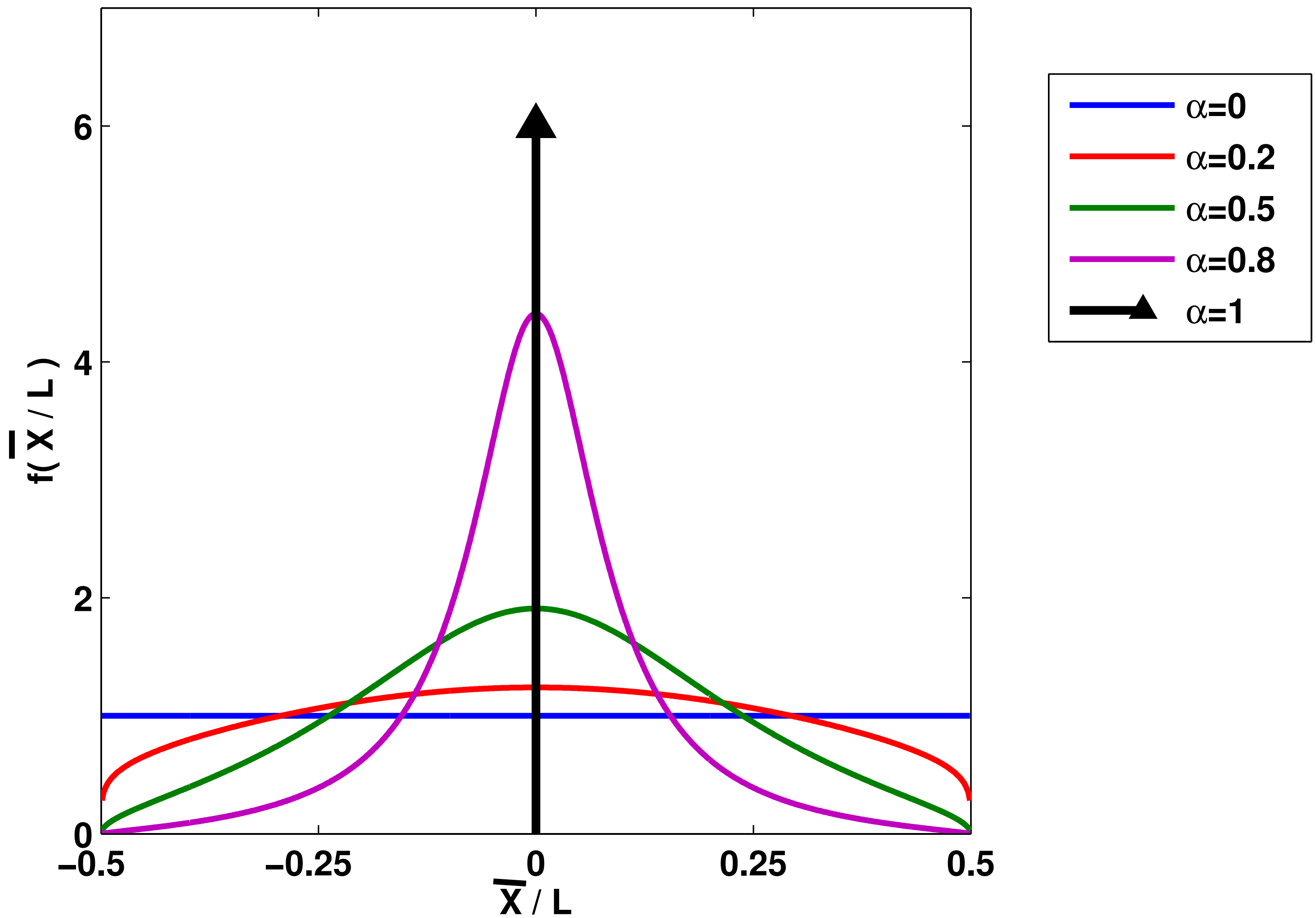
Ergodicity if $\alpha \to 1$

$$f_{\alpha=1}\left(\overline{\mathcal{O}}\right) = \delta\left(\overline{\mathcal{O}} - \langle\mathcal{O}\rangle\right).$$

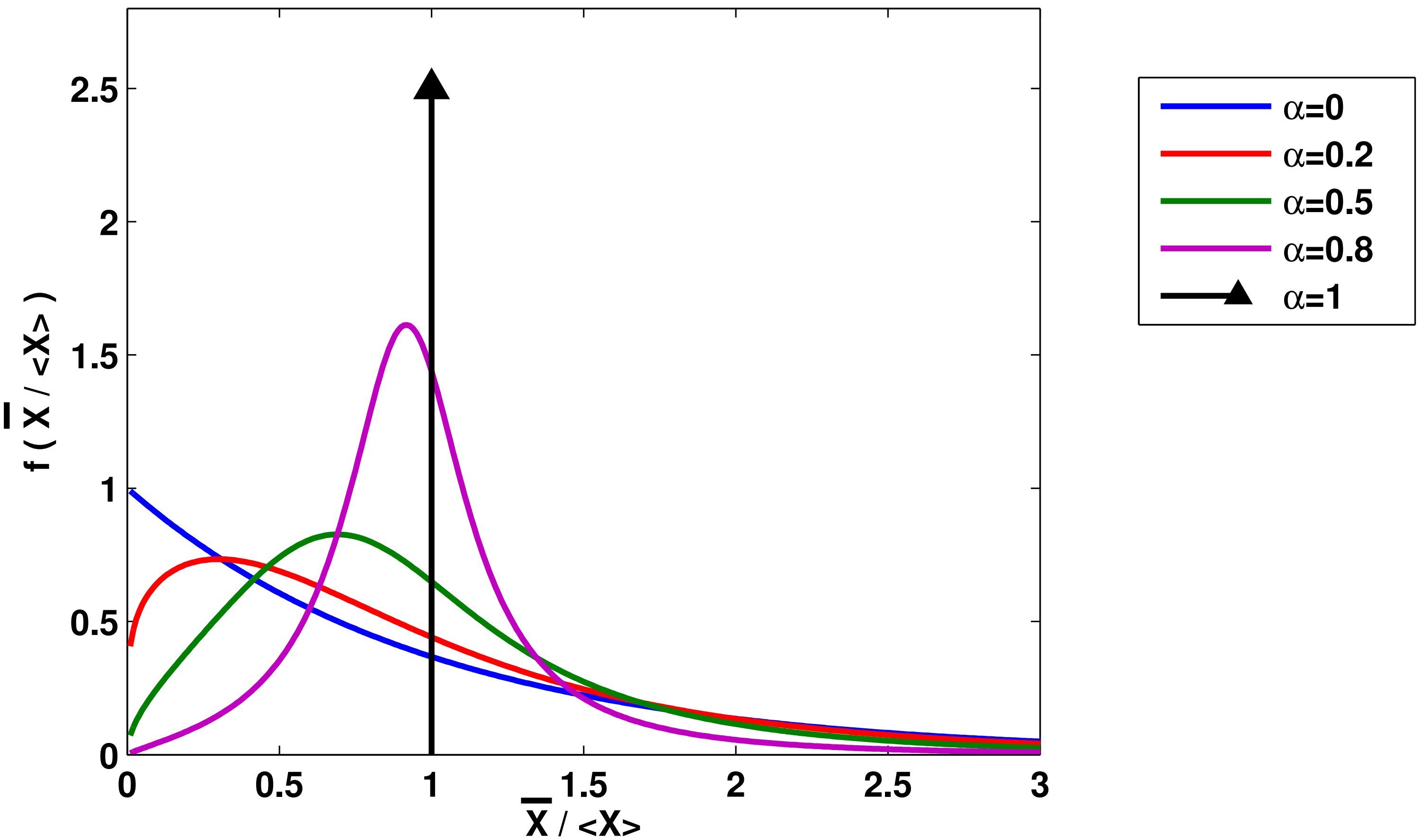
Localization when $\alpha \to 0$

$$\lim_{\alpha\to 0} f_\alpha\left(\overline{\mathcal{O}}\right) = \sum_{x=1}^{L} P_x^{eq}\delta\left(\overline{\mathcal{O}} - \mathcal{O}_x\right).$$

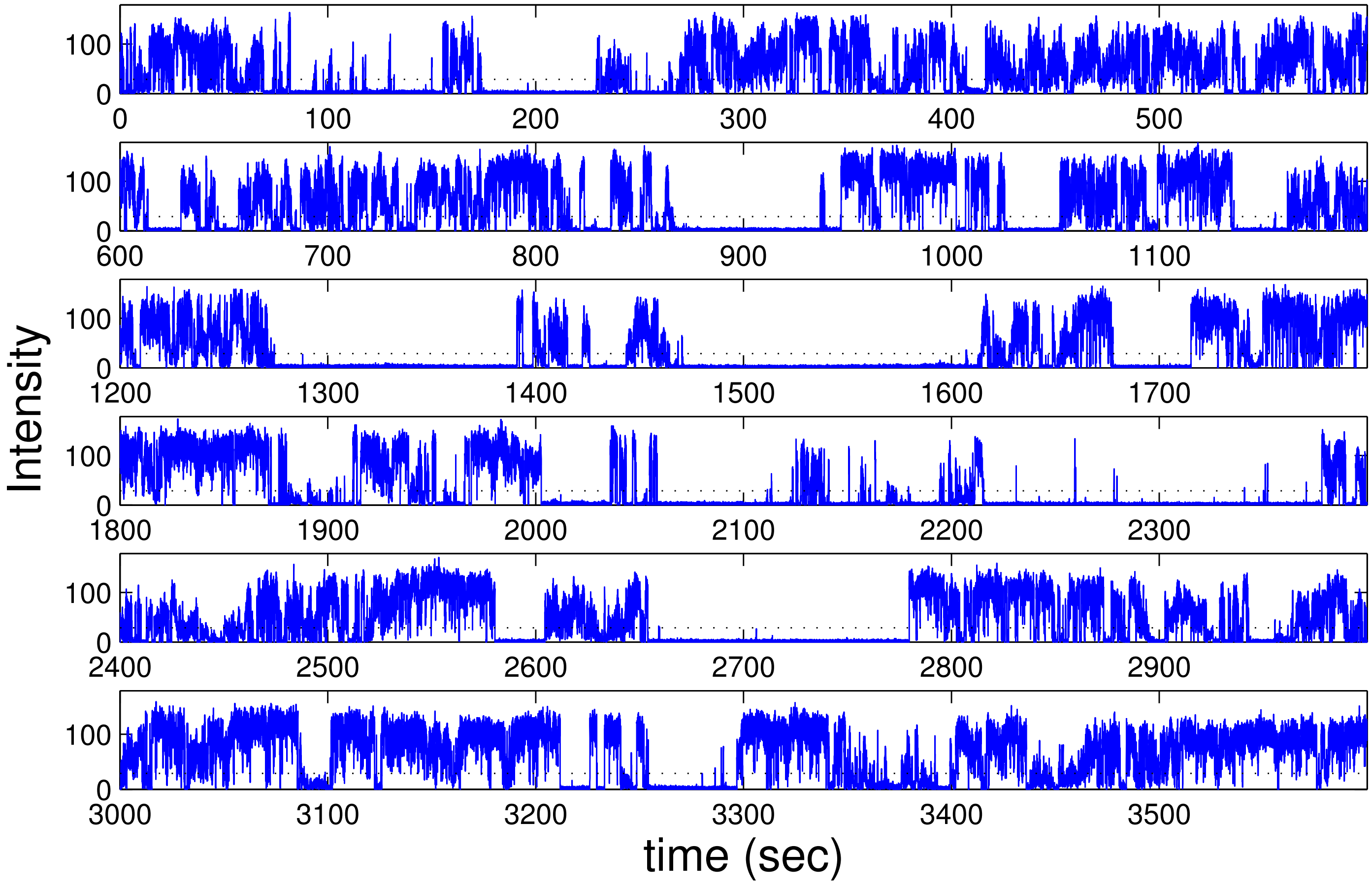
PDF of $\overline{X}$ UNBIASED CTRW



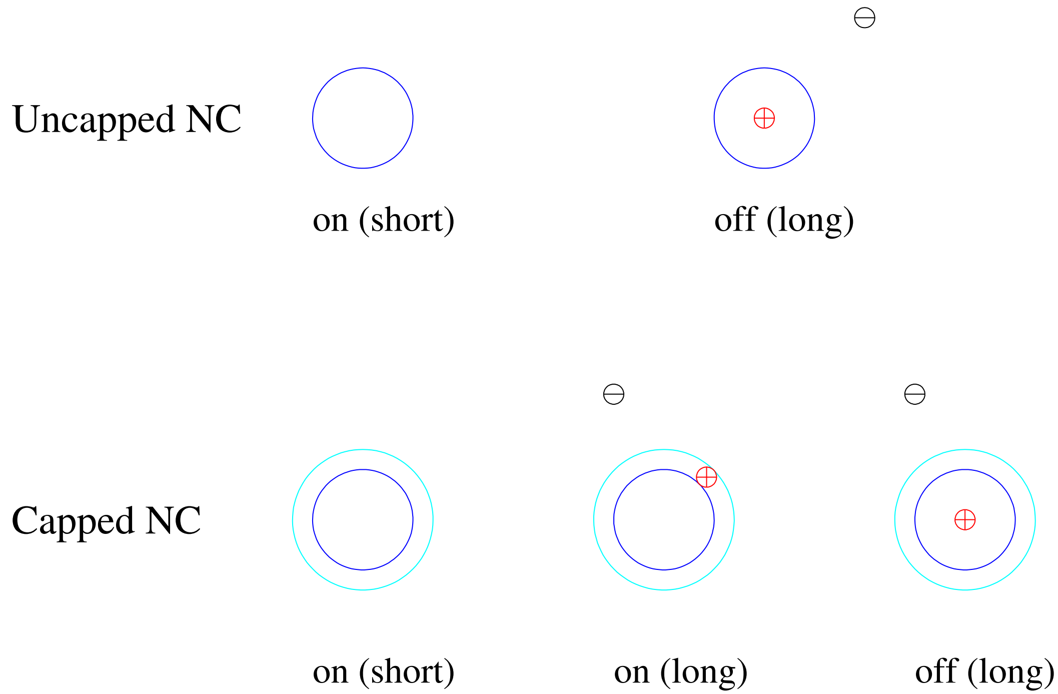
PDF of $\overline{X}$ BIASED CTRW



Blinking Nano Crystals



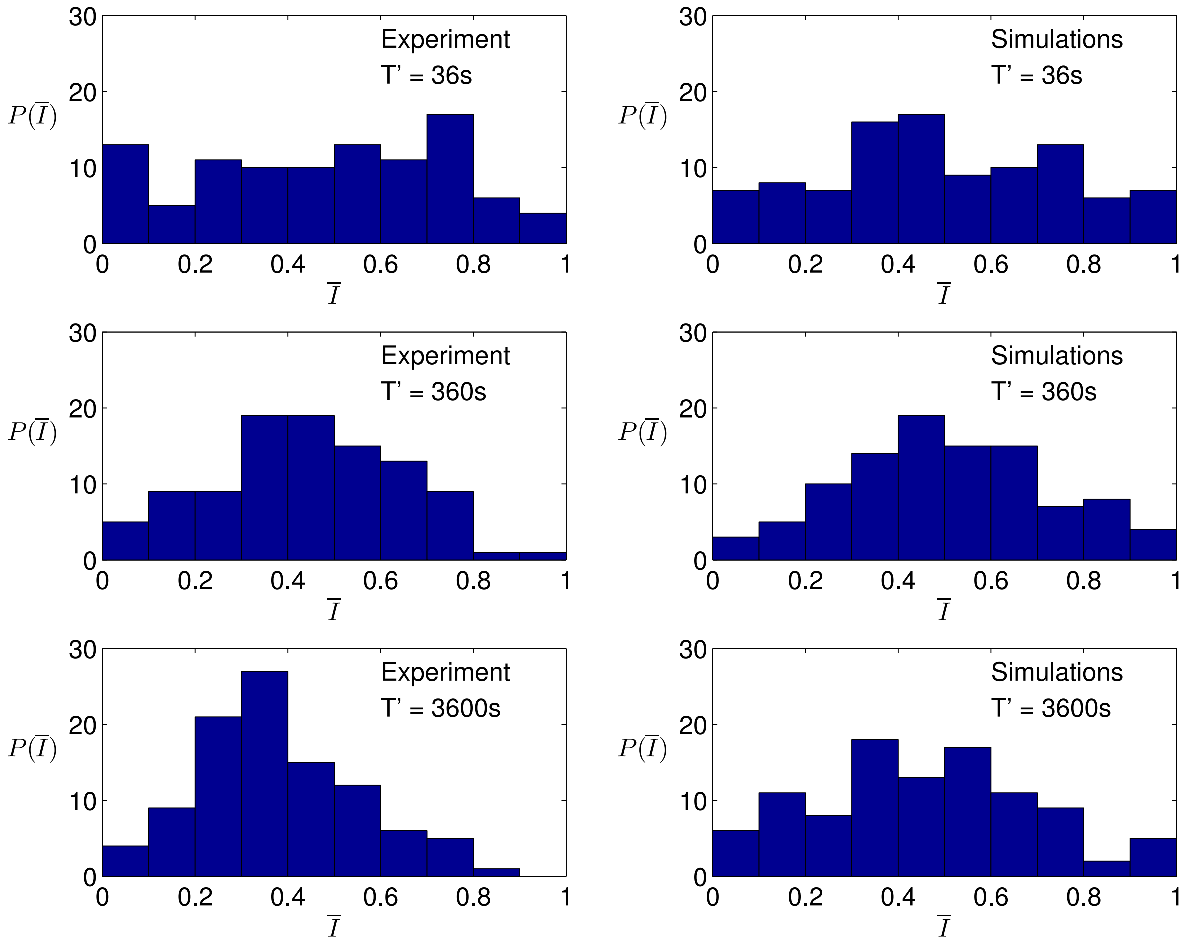
Group	Material	Nu.	Radii	T	α_{on}	α_{off}
Dahan	CdSe-ZnS	215	1.8nm	300 K	$0.58(0.17)$	$0.48(0.15)$
Orrit	CdS		2.85	1.2	EXP	$0.65(0.2)$
Bawendi	CdTe....	200	1.5	$10-300$	$0.5(0.1)$	$0.5(0.1)$
Kuno	CdSe-ZnS	300	2.7	300	$0.8-1.0$	0.5
Cichos	Si			300	$0.8-1.0$	0.5
Ha	CdSe(coat)			300	Exp?	1



Efros, Orrit, Onsager, Hong-Noolandi

$$r_{Ons} = \frac{e^2}{k_b T \epsilon} \simeq 7\text{nm} \quad (1)$$

Distribution of time averaged intensity $\overline{\mathcal{I}}$



CdSe-ZnS NC. Margolin, Kuno, Barkai (2006)

Summary

Boltzmann--Gibbs	WEB
$\alpha = 1$	$\alpha < 1$
normal diffusion	anomalous diffusion $\langle r^2 \rangle \sim t^\alpha$
Gaussian	Levy -- Lamperti
$f_1\left(\overline{\mathcal{O}}\right) = \delta\left[\overline{\mathcal{O}} - \langle \mathcal{O} \rangle\right]$	$f_\alpha\left(\overline{\mathcal{O}}\right) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \text{Im} \frac{\sum_{x=1}^{L} P_x^{eq} \left(\overline{\mathcal{O}} - \mathcal{O}_x + i\epsilon\right)^{\alpha-1}}{\sum_{x=1}^{L} P_x^{eq} \left(\overline{\mathcal{O}} - \mathcal{O}_x + i\epsilon\right)^{\alpha}}.$

Refs.

- Margolin, Barkai **Phys. Rev. Lett. 90**, 104101 (2005).
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- Margolin, Kuno, Protasenko, Barkai **Adv. Chem. Phys 133** 327 (2006).
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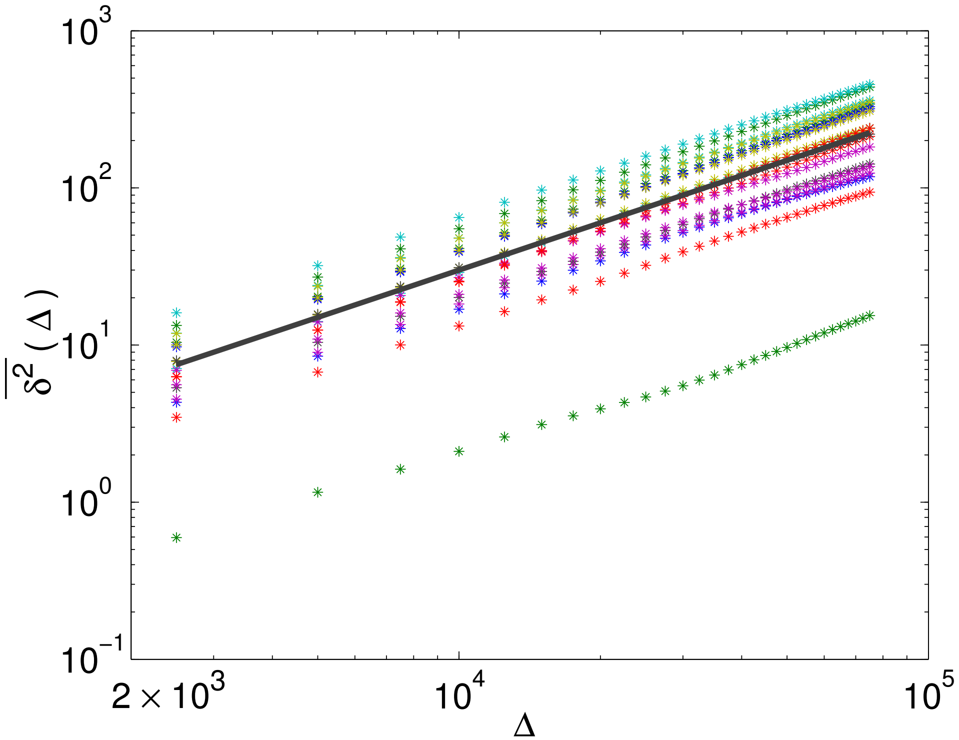
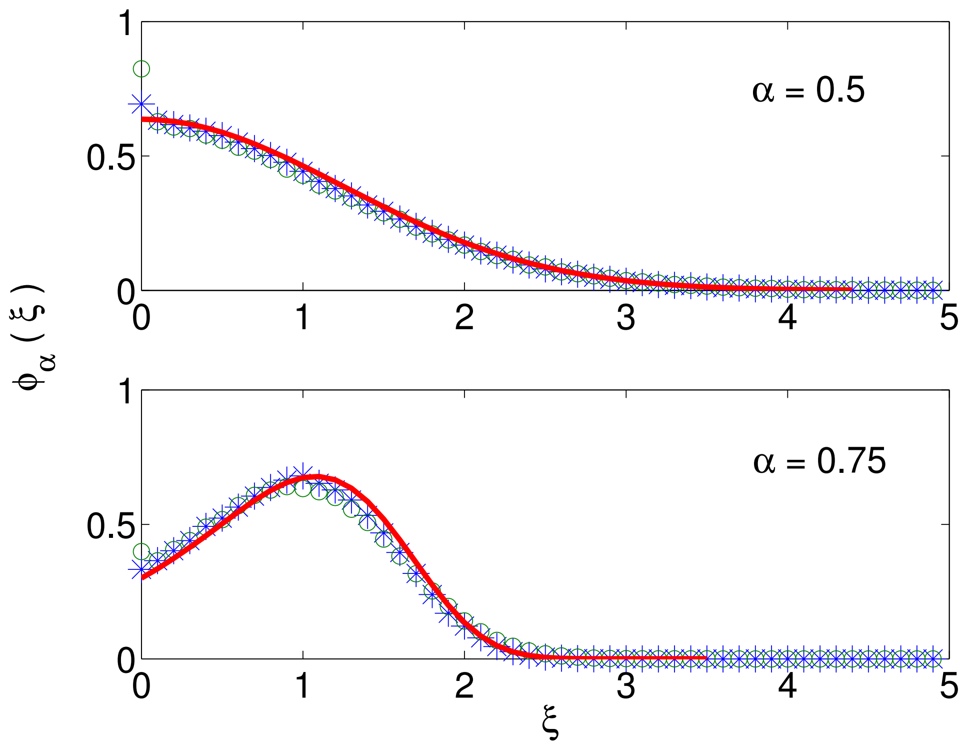
- NSF, PRF, ISF, ND, Complexity Center Jerusalem.

Random Time-Scale Invariant Diffusion Constant



$$\overline{\delta^2}\,(\Delta,t) = \frac{\int_0^{t-\Delta}\left[x(t'+\Delta)-x(t')\right]^2 dt'}{t-\Delta}$$

- $\overline{\delta^2} \sim N$
- $\xi = \overline{\delta^2} / \langle \overline{\delta^2} \rangle$



$$\lim_{t\to\infty} \phi_\alpha(\xi) = \frac{\Gamma^{1/\alpha}(1+\alpha)}{\alpha \xi^{1+1/\alpha}} l_\alpha \left[\frac{\Gamma^{1/\alpha}(1+\alpha)}{\xi^{1/\alpha}} \right].$$